

## Spectral properties of Bunimovich mushroom billiards

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Properties of a quantum mushroom billiard in the form of a superconducting microwave resonator have been investigated. They reveal unexpected nonuniversal features such as, e.g., a supershell effect in the level density and a dip in the nearest-neighbor spacing distribution. Theoretical predictions for the quantum properties of mixed systems rely on the sharp separability of phase space—an unusual property met by mushroom billiards. We however find deviations which are ascribed to the presence of dynamic tunneling.

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Billiards play a central role in the investigation of systems with regular, chaotic, or mixed dynamics [1]. When quantum chaos established itself as a new field, billiards were used as a paradigm for both theoretical and experimental research [2]. The billiard considered in this Rapid Communication is from the family of mushroom billiards suggested by Bunimovich [3] as a generalization of the stadium billiard [4]. Compared to conventional mixed systems they have the particular property that their phase space is sharply divided into one regular island and the chaotic sea, whereas usually the islands of regularity are typically surrounded by a layer of infinitely many islands. In the simplest case they consist of a semicircular hat with a symmetrically attached rectangular stem as shown in Fig. 1(a). All regular orbits of mushroom billiards are orbits of the semicircle billiard with a conserved angular momentum which stay in the hat forever. Orbits of particles with the same angular momentum form a semicircular caustic. There is one critical caustic, which rigorously separates the orbits into regular and chaotic ones; its radius  $r_c$  equals half the width of the stem [dotted line in Fig. 1(a)]. Particles moving in the hat with a larger or equal caustic stay there forever, whereas those with a smaller one eventually enter the stem and therefore are chaotic. To our knowledge this clear separation of the phase space has been found before only for classical maps [5]. Due to this unusual feature of the phase space, classical mushroom billiards are of interest with respect to different aspects [6]. How becomes the separability of the classical phase space visible in properties of the corresponding quantum system? This question motivated us to investigate a quantum mushroom billiard experimentally using the analogy between quantum and microwave billiards [7,8].

The geometry of the quantum billiard we investigated is shown in Fig. 1(b). To avoid the effects induced by the superposition of two parity classes, we used a desymmetrized mushroom billiard—i.e., one with a quarter-circle for the hat. Moreover, the stem is chosen triangular instead of rectangular in order to eliminate bouncing ball orbits as seen, e.g., in the stadium billiard [8]. The stem width is two-thirds of the radius  $R$  of the hat, and its inner angle equals  $45^\circ$ . We verified with a rigorous analysis [3] that the classical phase space is still sharply divided into a regular and a chaotic part. The invariant measure  $q_c$  of the chaotic part is 82.9% of the phase-space volume. The eigenvalues of the quantum mushroom billiard were measured with a flat, cylindrical microwave cavity of lead-plated copper [Fig. 1(c)] of 5 mm height. The

hat has a radius of  $R=0.24$  m. In order to obtain a large and reliable set of resonance frequencies we performed the measurements at 4.2 K, where the cavity is superconducting [8,9]. Using a vectorial network analyzer (VNA) we collected complex transmission spectra with six different antennae up to 22 GHz with a sample rate of 100 kHz. The antenna positions are distributed over the whole area of the billiard [Fig. 1(b)]. Figure 2 shows a part of the measured transmission spectra for five different antenna combinations. The spectra have the unusual property that they exhibit sequences of resonances, which are separated by large gaps with no resonances (arrows in Fig. 2). This bunching effect indicates that there are two well-distinguishable frequency scales. It will be discussed in detail below.

A set of 938 resonance frequencies was obtained from the spectra in agreement with the expectation from Weyl's formula [10]. On the basis of this large data set we first considered conventional statistics and compared them to theories applicable to the quantum spectra of mixed systems, such as, e.g., the Berry-Robnik statistics [11]. First we computed the number of levels  $N(f)$ —i.e., the integrated resonance density  $\rho(f)$ —and determined its smooth part  $N_{\text{Weyl}}(f)$ . The resulting fluctuating part  $N_{\text{fluc}}(f) = N(f) - N_{\text{Weyl}}(f)$  is shown in Fig. 3. The most striking feature is a beating pattern caused by the superposition of two oscillations, which in fact reflects the bunching effect observed in Fig. 2. In the stadium billiard [8] oscillations are caused by the so-called bouncing ball orbits. To find the origin of the beating observed in Fig. 3, we computed the length spectrum  $|\tilde{\rho}_{\text{fluc}}(x)|$  shown in Fig. 4(a), which has peaks at lengths  $x$  of periodic orbits. Here,  $\tilde{\rho}_{\text{fluc}}(x)$  is the Fourier transform of the fluctuating part of the resonance

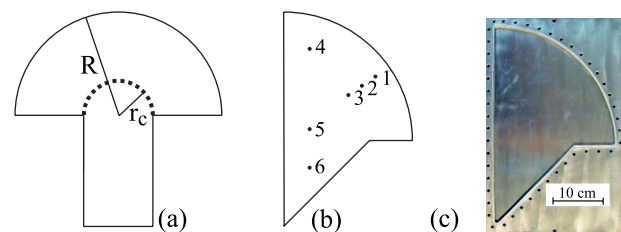


FIG. 1. (Color online) (a) Typical shape of a mushroom billiard with the critical caustic shown as a dotted line, (b) desymmetrized version (quarter-circle as hat) with triangular stem, and (c) photograph of the corresponding experimental microwave resonator. The positions of the six antennae in the experiment are indicated in (b).

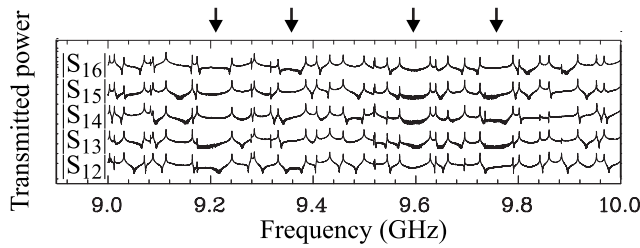


FIG. 2. Part of five transmission spectra between 9 and 10 GHz, measured with the microwave resonator shown in Fig. 1(c). The curves labeled with  $|S_{1j}|$ ,  $j=2, \dots, 6$ , are obtained with the antennae 1 and  $j$  whose positions are indicated in Fig. 1(b). The spectra have been displaced from each other along the  $y$  axis for illustration and are plotted in a logarithmic scale. One notices a bunching behavior—i.e., large gaps marked by arrows—between stretches of resonances.

density  $\rho_{\text{fluc}}(k)$  as a function of wave number  $k=2\pi f/c$ , where  $c$  is the velocity of light. In order to get information on the nature of the periodic orbits related to the peaks in the length spectrum, we also computed the latter for the regular orbits in the hat [8]. For this purpose we first determined the eigenvalues and eigenfunctions of a quarter-circle billiard of the same radius as the hat. These are indexed by a radial quantum number  $n$  and an angular momentum one  $m$ . The length spectrum in Fig. 4(b) has been obtained by considering only eigenvalues with eigenfunctions, which are localized between the critical caustic [Fig. 1(a)] and the circular boundary—that is, eigenfunctions with a sufficiently large  $m$ . By comparison of Fig. 4(a) with Fig. 4(b) we see that all peaks of regular orbits in the hat of the mushroom billiard are reproduced. The remaining peaks in the length spectrum are thus associated with chaotic orbits. Around 0.7 m we observe a pair of closely lying dominant peaks. They correspond to the shortest regular periodic orbits. After subtracting their contribution to  $N_{\text{fluc}}(f)$  the beating vanished. This cause of beatings in circular or spherical geometries by short dominant orbits of approximately the same length has already been observed in metal clusters [12] and termed the supershell effect. It has also been stressed in nuclei [13], nanowires [14], and recently in trapped dilute Fermi gases [15]. In mushroom billiards it is induced by regular orbits coexisting with chaotic ones whose measure is determined by the depth of the stem.

Next we consider the nearest-neighbor spacing distribu-

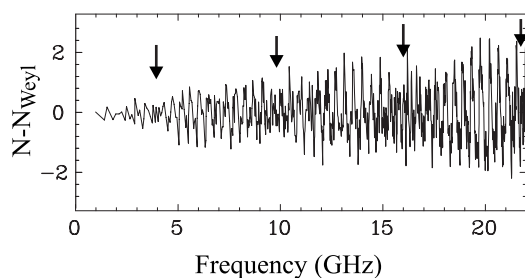


FIG. 3. Fluctuating part of the number of resonances below a given frequency  $f$ . The curve oscillates and shows a beating behavior with some noise at the nodes marked by arrows.

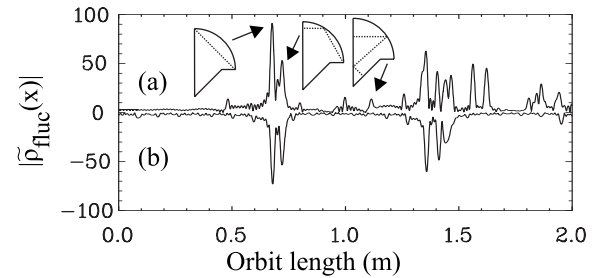


FIG. 4. (a) Experimental length spectrum of periodic orbits of the mushroom billiard. The two peaks of lengths of about 0.7 m are due to two regular orbits in the hat of the mushroom, and that at length 1.12 m is due to a chaotic one. (b) The computed length spectrum of a quarter-circle billiard is shown flipped upside down, where only eigenvalues were considered which correspond to angular momenta larger than the critical one.

tion (NND). Figure 5(a) shows the experimental NND with that for Poisson statistics and a Gaussian orthogonal ensemble (GOE) describing regular and chaotic systems, respectively. As to be expected the experimental NND coincides with neither of them. A comparison of the NND with the one derived by Berry and Robnik [11] for mixed systems yields good agreement for  $q_c=82.9\%$  as given above. Yet, even though the requirements for its applicability are best met with mushroom billiards, most surprisingly the agreement is not better than for other mixed systems. Especially, for small spacings  $s$  deviations are visible in Fig. 5(a). This can be attributed to dynamic tunneling between classically separated regions of phase space in the quantum regime [16–18]. In addition, we observe a peculiarity in the shape of the NND. Namely, the experimental histogram resulting from a fairly small binwidth exhibits a profound dip at  $s=0.7$ . This dip is statistically significant, and it is present also in NND's derived from different subsections of the spectrum. When including the contribution of the pair of dominant short periodic orbits to  $N_{\text{fluc}}(f)$  in the usual spectral unfolding procedure [8], the dip is no longer present [Fig. 5(b)]. It should be noted that this result is contrary to the general belief that only long periodic orbits control short-range spectral properties of a quantum system.

While in the semiclassical limit the eigenvalue spectrum

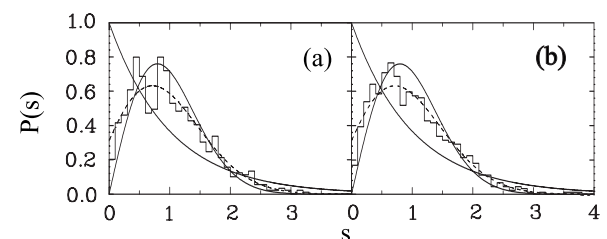


FIG. 5. (a) Nearest-neighbor spacing distribution of the mushroom billiard together with those for Poisson statistics and the GOE (solid lines) expected for systems with regular and chaotic dynamics, respectively. Note the dip at  $s=0.7$ . It vanishes (b) when the contribution of the dominant periodic orbits in Fig. 4 to the resonance density is subtracted. The dashed line in both (a) and (b) shows the Berry-Robnik prediction for a mixed system with  $q_c=82.9\%$  chaos.

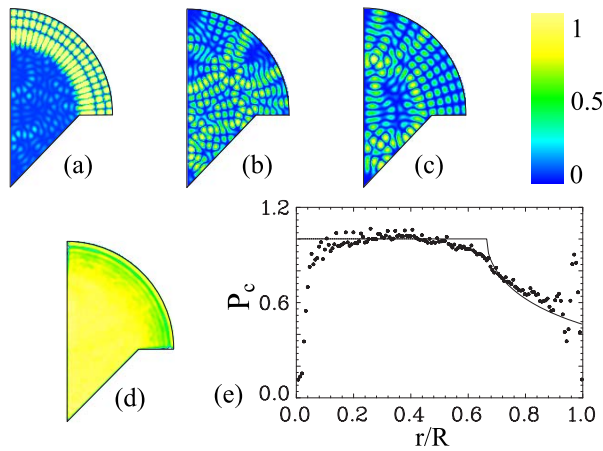


FIG. 6. (Color online) Measured intensity distributions. The eigenfunctions are usually either regular (a) or chaotic (b) and correspond to eigenfunctions of a quarter-circle billiard in the former case. We also found some mixed eigenfunctions like the one in (c). In (d) the averaged distribution of 239 chaotic eigenfunctions is shown. The radial intensity profile in the hat is plotted in (e) between the quarter-circle center and the circular boundary (points), and compared to the classical probability of finding chaotic orbits (solid line). The decrease beyond the critical caustic at  $r/R=2/3$  is revealed in the measurements, and deviations at the boundaries and at the knee are due to quantum effects.

of a mixed system consists of regular eigenvalues with eigenfunctions localized on the regular islands—i.e., for the desymmetrized mushroom billiard those of a quarter-circle billiard with eigenfunctions localized in the hat—and of chaotic ones with eigenfunctions distributed over the whole billiard [19], this must not be true anymore [16,17] in the quantum limit. However, since for mushroom billiards there is no layer of islands between the regular and chaotic parts of the phase space, their eigenstates should be classifiable as regular or chaotic [16,20] even in the quantum regime. In order to investigate this further, we measured the electric field strength intensity—i.e., squared eigenfunctions of an enlarged copy of the mushroom billiard—for several resonance frequencies using the perturbing bead method at room temperature [21]. An example for a squared regular eigenfunction is shown in Fig. 6(a). It is very similar to that of the quarter-circle with  $n=3$  and  $m=44$ . A chaotic eigenfunction is plotted in Fig. 6(b). However, for such eigenfunctions we even find traces of regularity in the field pattern in the hat, and for the regular eigenfunctions the intensity in the stem is nonvanishing. This indicates that there is a dynamic tunneling [18], such that the classification into regular and chaotic eigenstates is only asymptotically correct. Indeed, we also observed some rare mixed eigenfunctions whose intensity is equally distributed over the whole billiard area, while the field pattern in the hat is very similar to that of a regular eigenfunction [Fig. 6(c)]. In Fig. 6(d) the averaged intensity distribution of 239 chaotic eigenfunctions is shown. Its classical counterpart is the probability  $P_C$  to find chaotic orbits at a certain position in the billiard. It is constant as in completely chaotic systems in that part of the billiard which is accessible to chaotic orbits only—i.e., in the stem and in the hat for  $0 < r < r_c$ , where  $r$  is the distance from the circle

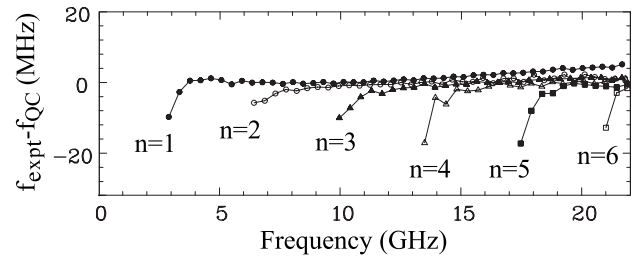


FIG. 7. Deviations of the experimental eigenfrequencies of the mushroom billiard from those of a quarter-circle billiard versus frequency. The symbols (points, triangles, squares) refer to regular states. Eigenvalues with equal radial quantum number  $n$  are connected by lines. For a given  $n$  there are almost no deviations for high frequencies where the eigenfunctions are localized close to the circular boundary. For smaller frequencies, the influence of the stem leads to increasing deviations pointing to the importance of dynamic tunneling.

center and  $r_c$  is the radius of the critical caustic. Interestingly, for  $r_c < r < R$  it decreases as  $\arcsin(r_c/r)$  [3], which is a consequence of equipartition in the mixed phase space. The radial profile of  $P_C$  is compared to the experimental results in (e)—revealing the decrease of intensity in the hat. The abrupt change of the distribution when crossing the critical caustic is smeared out due to the finite wavelength. At the boundary of the billiard quantum effects [22] also lead to deviations from the classical behavior.

Finally, for a fixed  $n$ , the eigenvalues of the quarter-circle become asymptotically equidistant for large frequencies. Thus regular modes in the superconducting mushroom billiard were obtained by identifying chains of equidistant eigenvalues in the measured spectrum and comparing them to the computed ones of the corresponding circle billiard. We found regular (chaotic) periodic orbits in the length spectrum of the regular (chaotic) eigenvalues and, though strongly suppressed, also chaotic (regular) ones appear. This shows again that there is an interaction via dynamic tunneling between the regular and chaotic parts of the classical phase space. The deviations of the measured regular eigenvalues of the mushroom billiard from the computed eigenvalues of the quarter-circle billiard shown in Fig. 7 quantify the dynamic tunneling. Most interestingly they vary unidirectionally. Indeed, for all families of equal radial quantum number  $n$  the deviations are close to zero for large frequencies—i.e., large values of  $m$ —where the eigenfunctions of the quarter-circle billiard are localized close to the circular boundary and therefore almost not influenced by the stem. When approaching the frequency corresponding to the critical angular momentum from above, the deviations increase as then the eigenfunctions are localized closer and closer to the critical caustic and the field distribution is increasingly distorted toward the stem as can be seen in the intensity plot in Fig. 6(a). This is exactly the dynamic tunneling. There is no general quantitative explanation for the measured deviations yet. Our results should, however, inspire forthcoming theoretical considerations of dynamic tunneling.

In summary we have shown that the spectral properties of mushroom billiards are significantly affected by the shortest regular orbits in the hat. They cause a supershell structure in

the level density. As compared to other systems exhibiting the supershell effect, in mushroom billiards the degree of chaoticity can be tuned by varying the depth of the stem, such that they allow the study of supershell effects for an arbitrary degree of chaos. Surprisingly, the shortest (not long) periodic orbits lead to a substructure in the NND. The eigenstates of the mushroom billiard may be separated into regular, chaotic, or—though rare—mixed ones. That the latter are rare and the behavior of the averaged intensity distribution of the chaotic eigenfunctions in the mushroom hat are manifestations of the separability of the classical phase space in the spectral properties of the quantum billiard. Still, dy-

namic tunneling is present and can be observed, e.g., in the field distributions, in the spectral properties, and in the deviations of the regular eigenvalues from those of the corresponding circle billiard. Since the tunneling barrier is of a simple structure for mushroom billiards, they are convenient systems for the study of the dynamic tunneling process.

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